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Linearized Boltzmann Equations I: Representation of Solutions

Henryk Gzyl[1]

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In this paper we obtain a probabilistic representation of the solutions of a linearized Boltzmann equation. By making use of dual Markov processes we extend Pinsky's results to the case where there is a gradient force field present.

KEY WORDS: Linearized Boltzmann equation; dual Markov processes; multiplicative functionals.

1. INTRODUCTION AND PRELIMINARIES

In this paper we obtain a probabilistic representation of the solution of a linearized Boltzmann equation of the type

$$\begin{aligned} \frac{\partial \rho}{\partial t}(x,v,t) + v\cdot\frac{\partial}{\partial x}\rho(x,v,t) + \frac{F(x)}{m}\cdot\frac{\partial}{\partial v}\rho(x,v,t) \\ = \lambda(v)\int \Pi(v',v)\big[k(v,v)\rho(x,v',t) - \rho(x,v,t)\big]\,dv' \\ \lim \rho(x,v,t) = \rho_0(x,v) \qquad \text{as} \quad t\downarrow 0 \end{aligned} \tag{1.1}$$

where $(x,v)\in\mathbb{R}^6$, $t>0$, dots indicate scalar products, and $\partial/\partial x$, $\partial/\partial v$ are the usual gradients. The rest of the terms will be explained below. In this way we extend Pinsky's result[1] to the case where a force field $F(x)$ is present.

Our method of solution will rely on the notion of dual Markov process (a generalization of the notion of time-reversed process) and the proof

[1] Escuela de Fisica y Matemáticas, Facultad de Ciencias, Universidad Central de Venezuela, Caracas.

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follows from a simple renewal equation type argument. This also gives an alternative derivation of the results in Ref. 1 to the case $F(x) \neq 0$. This is developed in Section 2. After obtaining the probabilistic representation of the solution to (1.1) we examine some of its elementary properties, and we leave for papers II and III of this series a variety of limits theorems, perturbation expansions, boundary-value problems, and applications.

In what follows we shall present all necessary concepts as simply as possible, giving only the essentials needed for the solution of (1.1) and not getting bogged down in—very hard—technical issues. Stating all preliminaries will take longer than obtaining the solution to (1.1), and we hope the reader finds it worthwhile.

We shall say that two processes $(Z_t)_{t>0}$ and $(\hat{Z}_t)_{t>0}$ on the same state space (S, Σ), Σ denoting the Borel sets of S, and semigroups P_t and $\hat{P}_t$, respectively, are in duality with respect to a measure $m(dz)$ on S' if

$$\int \hat{P}_t f(z) g(z) m(dz) = \int f(z) P_t g(z) m(dz) \tag{1.2}$$

for every pair of functions f, g [usually in some Banach space of functions contained in $L^1(m) \cap L^2(m)$]. For the general definition the very interested reader can check with Ref. 2, and for a connection with time reversal in Markov processes and applications, Refs. 3 and 4.

For our needs two very simple cases of processes in duality will suffice. These are presented as separate processes which will be put together at the end, and at the same time we shall establish the notation for Section 2.

Example 1. Let ϕ_t and $\tilde{\phi}_t$ be the flows in $\mathbb{R}^6$ associated to the solutions of the systems

$$\frac{dX}{dt} = V, \qquad \frac{dV}{dt} = \frac{F(x)}{m} \tag{1.3a}$$

$$\frac{d\tilde{X}_t}{dt} = -\tilde{V}, \qquad \frac{d\tilde{V}}{dt} = -\frac{F(\tilde{X})}{m} \tag{1.3b}$$

where $F(x) = -\nabla U(x)$ is such that the solutions through every (x, v) exist globally and are unique. For any continuous function $f(x, v)$ on $\mathbb{R}^6$ we define

$$T_t f(x, v) = f(\phi_t(x, v)), \qquad \tilde{T}_t f(x, v) = f(\tilde{\phi}_t(x, v)) \tag{1.4}$$

where $\phi_t(x, v)$ and $\tilde{\phi}_t(x, v)$ are the solutions to (1.3a) and (1.3b) which at $t = 0$ pass through (x, v). Also $\tilde{\phi}_t(x, v) = \phi_{-t}(x, v)$ and since we assume the systems to be Hamiltonian, ϕ_t is a measure-preserving flow on $\mathbb{R}_6$. Therefore

$$\begin{aligned}\int f(x,v)T_t g(x,v)\,dx\,dv &= \int f(x,v)g(\phi_t(x,v))\,dx\,dv \\ &= \int f(\phi_{-t}(x,v))g(x,v)\,dx\,dv \\ &= \int f(\tilde{\phi}_t(x,v))g(x,v)\,dx\,dv \\ &= \int \tilde{T}_t f(x,v)g(x,v)\,dx\,dv\end{aligned}$$

This asserts that the processes $Z_t = (X_t, V_t)$ and $\tilde{Z}_t = (\tilde{X}_t, \tilde{V}_t) = (X_{-t}, V_{-t})$ are dual. Observe that though $\tilde{Z}_t$ is the time-reversed process of Z_t, the equations of motion in Hamiltonian form are not invariant under time reversal.

It is also very easy to check that for continuously differentiable $f(x,v)$ on $\mathbb{R}^6$

$$\frac{dT_t f(x,v)}{dt} = G_0 T_t f(x,v), \qquad \frac{d\tilde{T}_t}{dt} = -G_0\tilde{T}_t f(x,v)$$

where

$$G_0 = v\cdot\frac{\partial}{\partial x} + \frac{F(x)}{m}\cdot\frac{\partial}{\partial v} \tag{1.5}$$

Example 2. Let V_t now denote a pure jump Markov process (the generalization of birth and death process, see Ref. 2) on $\mathbb{R}^3$ whose infinitesimal generator is given by

$$G_1 f(v) = \lambda(v)\int \Pi(v,v')[f(v') - f(v)]\,dv' \tag{1.6}$$

where $\lambda(v)^{-1}$ is the average time the process stays at v before jumping away and $\Pi(v,v')\,dv'$ is the probability that a jump from v ends up in dv' around v'.

Naturally, $\int\Pi(v,v')\,dv' = 1$ for all v, and we shall assume that

$$\int \Pi(v,v')\,dv = 1 \qquad \text{for all} \quad v' \tag{1.7}$$

Now, put $\tilde{\Pi}(v,v') = \Pi(v',v)$ and $\tilde{\lambda}(v) = \lambda(v)$; define

$$\tilde{G}_1 f(v) = \tilde{\lambda}(v)\int \tilde{\Pi}(v,v')[f(v') - f(v)]\,dv' \tag{1.8}$$

It is easy to check (leaving aside domain questions) that if S_t and $\tilde{S}_t$ are the semigroups with generators G_1 and $\tilde{G}_1$, i.e., if

$$\frac{dS_t}{dt} = G_1 S_t, \qquad \frac{d\tilde{S}_t}{dt} = \tilde{G}_1 \tilde{S}_t$$

then

$$\int f(v) S_t g(v)\, dv = \int \tilde{S}_t f(v) g(v)\, dv$$

Now, if we put these two sets of generators together and define

$$Gf(x,v) = G_0 f(x,v) + G_1 f(x,v) \tag{1.9a}$$

$$\tilde{G}f(x,v) = -G_0 f(x,v) + \tilde{G}_1 f(x,v) \tag{1.9b}$$

Of course, the left-hand sides are to be applied to differentiable functions such that the integrals in (1.6) and (1.8) are finite. There is a theory developed by Bass[5,6] of how to add jumps to a given process through modifications of their infinitesimal generators as in (1.9).

From now on $Z_t = (X_t, V_t)$ and $\tilde{Z}_t = (\tilde{X}_t, \tilde{V}_t)$ are processes on $\mathbb{R}^6$ whose infinitesimal generators are given by (1.9a) and (1.9b), respectively, and we shall denote by P_t and $\tilde{P}_t$ their respective semigroups; then from the duality of G and $\tilde{G}$ we can readily obtain that of P_t and $\tilde{P}_t$. The most simple-minded way is to use the exponential formulas for P_t and $\hat{P}_t$. We remark now that even though Z_t and $\tilde{Z}_t$ are dual, they are not the time reversal of each other any more.

Before giving the very elementary properties of Z_t (which are analogous for $\tilde{Z}_t$) let us mention that $P^{x,v}$ and $\tilde{P}^{x,v}$ will denote the measures on the space of trajectories (right continuous with left limits) on $\mathbb{R}^6$ constructed from P_t and $\tilde{P}_t$. By $E^{x,v}$ and $\tilde{E}^{x,v}$ we shall denote the expected (average) values with respect to $P^{x,v}$ and $\tilde{P}^{x,v}$, respectively.

The behavior of Z_t will be as follows: starting from (x, v), Z_t will equal $\phi_t(x,v) \equiv (x(t), v(t))$ up to a random time T_1 with distribution

$$P^{x,v}(T_1 > t) = \exp\left[-\int_0^t \lambda(v(s))\, ds \right] \tag{1.10}$$

at which the velocity has a discontinuity (receives a kick from the environment) according to

$$P^{x,v}(V(T_1) \epsilon dv' \mid V(T_1 -) = v) = \Pi(v,v')\, dv' \tag{1.11}$$

where

$$V(T_{\bar{1}}) = \lim_{S \uparrow T_1} V_S = \lim_{S \uparrow T} v(s) = v(T_{\bar{1}})$$

From T_1 on, and up to another T_2 independent of T_1 and distributed according to (1.10), it will evolve according to the flow ϕ_t with initial condition $(X(T_1), V(T_1))$, and at T_2 the velocity has another discontinuity distributed according to (1.11), etc., etc.

Also, there are finitely many jumps in every time interval. From (1.10) and (1.11) it follows that for any positive measurable function $g(x, v, v')$

$$E^{x,v}[\, g(X(T_1), V(T_{\bar{1}}), V(T_1)); T_1 < t\,]$$
$$= \int_0^t \lambda(v(s)) \exp\left[-\int_0^s \lambda(v(u))\, du\right] ds \int \Pi(v(s), v') g(x(s), v(s) v')\, dv' \tag{1.12}$$

We still need to define one more object. Let $k(v, v')$ be a continuous function on $\mathbb{R}^6$ such that $k(v, v) = 1$ for all v. Define

$$\tilde{m}_t = \prod_{s<t} k(V_{s-}, V_s), \qquad m_t = \prod_{s<t} k(V_s, V_{s-}) \tag{1.13}$$

We denote by Θ_t, $\tilde{\Theta}_t$ the time-shift operators, i.e., $Z_t \cdot \Theta_s = Z_{t+s}$ and $\tilde{Z}_t \cdot \tilde{\Theta}_s = \tilde{Z}_{t+s}$; then

$$m_{t+s} = m_t m_s \cdot \Theta_t, \qquad m_{t+s} = \tilde{m}_t \cdot \tilde{m}_s \cdot \Theta_t \tag{1.14}$$

i.e., m_t and $\tilde{m}_t$ are multiplicative functionals of Z_t and $\tilde{Z}_t$. If we require that, as a function of v, the following integrals

$$\int \tilde{\Pi}(v, v') k(v, v') g(v')\, dv', \qquad \Pi(v, v') k(v', v)\, g(v')\, dv'$$

are bounded, then $E^{x,v}|m_t|$, $\tilde{E}^{x,v}|m_t|$ will be finite for all t. What is more important for us here is that m_t and $\tilde{m}_t$ are in duality, i.e.,

$$\int f(x, y) E^{x,v}[\, m_t g(X_t, V_t)\,]\, dx\, dv = \int \tilde{E}^{x,v}[\, \tilde{m}_t g(\tilde{X}_t, \tilde{V}_t)\,] f(x, v)\, dx\, dv \tag{1.15}$$

for f and g in the appropriate class. This result can be found in Ref. 7.

2. SOLUTION OF EQ. (1.1)

With the notations introduced in Section 1 we can state our extension of Pinsky's result as follows:

Theorem 2.1. Let $\lambda(v)$, $\Pi(v, v')$ be such that the process $(\tilde{X}, \tilde{V})$ with generator $\tilde{G}$ given by (1.9b) is well defined and in duality with (X, V) with respect to the multiplicative functionals $m, \tilde{m}$ [i.e., (1.15) holds]. Let $\rho_0(x, v)$ be a bounded, continuously differentiable function in the domain of $\tilde{G}$. Then

$$\rho(x, v, t) = \tilde{E}^{x,v}[\, \tilde{m}_t \rho_0(\tilde{X}_t, \tilde{V}_t)\,] \tag{2.1}$$

is well defined, positive if k and ρ are positive, and satisfies

$$\begin{gathered}\frac{\partial\rho}{\partial t}+G_0\rho=\lambda(v)\int\Pi(v,v')\left[k(v,v')\rho(x,v',t)-(x,v,t)\right]dv' \\ \lim\rho(x,v,t)=\rho_0(x,v) \qquad \text{as} \quad t\downarrow 0\end{gathered} \tag{2.2}$$

Proof. We shall write a renewal equation for $\rho(x,v,t)$ and by differentiation we obtain (2.2). This is more related to the methods in Ref. 8 than to Ref. 1.

The limit behavior at $t=0$ follows from the continuity of k and of ρ_0 and the right continuity of $(\tilde{X}_t,\tilde{V}_t)$.

Let now $\tilde{T}=\inf\{t>0: \tilde{V}_t\neq\tilde{V}_{t-}\}$ be the time of the first discontinuity of $\tilde{V}_t$ (the first kick from the environment) and let us write

$$\rho(x,v,t)=E^{x,v}\left\{\tilde{m}_t\rho_0(\tilde{X}_t,\tilde{V}_t);T>t\right\}+\tilde{E}^{x,v}\left\{\tilde{m}_t\rho_0(\tilde{X}_t,\tilde{V}_t);\tilde{T}\leqslant t\right\}$$

and let us compute each term on the right-hand side separately. For the first term we can write

$$\tilde{E}^{x,v}\left\{\tilde{m}_t\tilde{\rho}_0(\tilde{X}_t,\tilde{V}_t);t<\tilde{T}\right\}=\rho_0(\tilde{\phi}_t(x,v))\exp\left(-\int_0^t\lambda(\tilde{v}(s))\,ds\right)$$

This is true because before the first jump of $\tilde{V}_t$, $\tilde{m}_t=1$, and $\rho_0(\tilde{X}_t,\tilde{V}_t)=\rho_0(\tilde{\phi}_t(x,v))$ and because of (1.10).

The second term yields

$$\begin{aligned}
&\tilde{E}^{x,v}\left\{\tilde{m}_t\rho_0(\tilde{X}_t,\tilde{V}_t);\tilde{T}\leqslant t\right\}\\
&\quad=\tilde{E}^{x,v}\left\{m_{\tilde{T}}\left(\tilde{m}_{t-\tilde{T}}\rho_0(\tilde{X}_{t-\tilde{T}},\tilde{V}_{t-\tilde{T}})\right)\cdot\tilde{\Theta}_{\tilde{T}};\tilde{T}\leqslant t\right\}\\
&\quad=\tilde{E}^{x,v}\left\{k(\tilde{V}(\tilde{T}-),\tilde{V}(\tilde{T}))\tilde{E}^{\tilde{X}(\tilde{T}),\tilde{V}(\tilde{T})}\left\{\tilde{m}_{t-\tilde{T}}\rho_0(\tilde{X}_{t-T},\tilde{V}_{t-\tilde{T}})\right\};\tilde{T}<t\right\}\\
&\quad=\tilde{E}^{x,v}\left[k(\tilde{V}(\tilde{T}-),\tilde{V}(T)\rho(\tilde{X}(\tilde{T}),\tilde{V}(\tilde{T}),t-\tilde{T}));\tilde{T}<t\right]\\
&\quad=\int_0^t\lambda(\tilde{v}(s))\exp\left[-\int_0^s\lambda(\tilde{v}(u))\,du\right]ds\\
&\qquad\times\int\Pi(\tilde{v}(s),v')k(\tilde{v}(s)v')\rho(\tilde{x}(s),v',t-s)\,dv'
\end{aligned}$$

where at the first step we used the multiplicative property at time $\tilde{T}$, at the second the Markov property of $(\tilde{X}_t,\tilde{V}_t)$, at the third the definition (2.1), and at the last step we made use of (1.12).

Adding up these two results we obtain an integral equation for $\rho(x,v,t)$ (of renewal type in the Markov processes jargon). Computing the derivative at $t=0$ is easy and since for fixed $g(x,v)$, $Q_tg(x,v)$

$= \tilde{E}^{x,v}\{\tilde{m}_t g(\tilde{X}_t, \tilde{V}_t)\}$ defines a homogeneous semigroup, it follows that, since (2.1) satisfies (2.3) at $t = 0$, it does so at any t.

In this setup, computing the time derivative of $Q_t g(x,v)$ is tantamount to computing an infinitesimal generator of "subordinated processes," which is a rather standard computation now using "stochastic calculus" as in Ref. 1.

Let us now comment on the most obvious properties of the representation (2.1).

First, note that if k is not identically 1, the normalization property $\int \rho_0(x,v)dx\,dv = 1$ is lost and we get instead, using (1.15),

$$\int \rho(x,v,t)\,dx\,dv = \int \tilde{Q}_t\rho_0(x,v)\,dx\,dv = \int \rho_0(x,v)\,Q_t 1(x,v)\,dx\,dv$$
$$= \int \rho_0(x,v)\,E^{x,v}\,\{m_t\}\,dx\,dv$$

When $k(v,v')$ is bounded, a computational analogous to the one in 1.3.2 of Ref. 8 would yield the exponential bound $|\int \rho(x,v,t)\,dx\,dv| < \exp tK$ for some appropriate K.

Also, the representation obtained would extend to the case where $\lambda = \lambda(x,v)$ and $\Pi = \Pi(x,v,v')$ are spatially inhomogeneous with just notational changes.

As we mentioned in the Introduction, this is to be continued with a study of a variety of limit theorems obtained from the representation of the solutions and some applications to boundary-value problems.

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